Parallel hp-FEM

hp-adaptive, hybrid-GMG, MatrixFree

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What is hp-adaptive FEM?

- Align mesh resolution with complexity of current solution

  h-adaptation: dynamic cell sizes

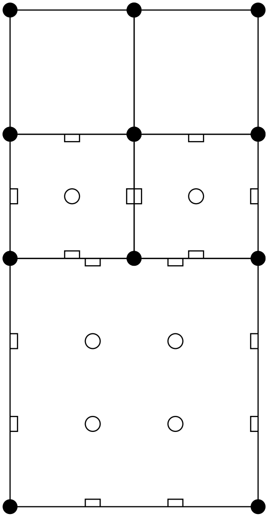
  good for irregular solutions

  p-adaptation: dynamic function spaces

  good for smooth solutions

- Combination of both possible

- *Algebraic* convergence with h-adaptation
- *Exponential* convergence possible with hp-adaptation



Timeline of hp-implementation in deal.II

2007/09/07	deal.II 6.0:	hp::DoFHandler, step-27
2009/04/27	deal.II 6.2:	FE_Nothing
2010/06/25	deal.II 6.3:	FE_Nothing
2011/10/09	deal.II 7.1:	step-46
2017/04/06	deal.II 8.5:	FESeries, step-27 rework

parallel

2018/05/11	deal.II 9.0:	hp::DoFHandler with shared::Triangulation
2019/05/21	deal.II 9.1:	hp::DoFHandler with distributed::Triangulation
2020/05/20	deal.II 9.2:	hp::Refinement, future FE indices
2021/06/01	deal.II 9.3:	MGTransferGlobalCoarsening, step-75, step-27 rework

How to use hp in deal.II?

- Enable hp-mode in `DoFHandler` with `hp::FECollection`
  - Note: `hp::DoFHandler` no longer required
- `active_fe_index` sets FE on each cell

- `future_fe_index` determines FE on each cell after refinement
- `hp::Refinement` namespace offers decision strategies
- Decision indicators via `SmoothnessEstimator`, `predict_error`

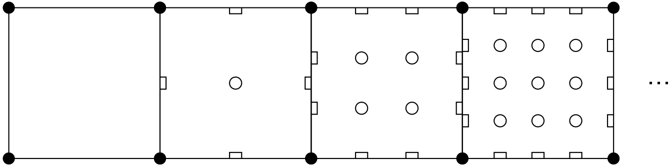
Serial example

`step-27` demonstrates a basic serial hp-application.

Load balancing

- Partition domain into subdomains for parallelization
- Balance workload on all subdomains

- Workload on each cell proportional to number of DoFs
- Requires *weighted repartitioning* for load balancing
  - `parallel::CellWeights`



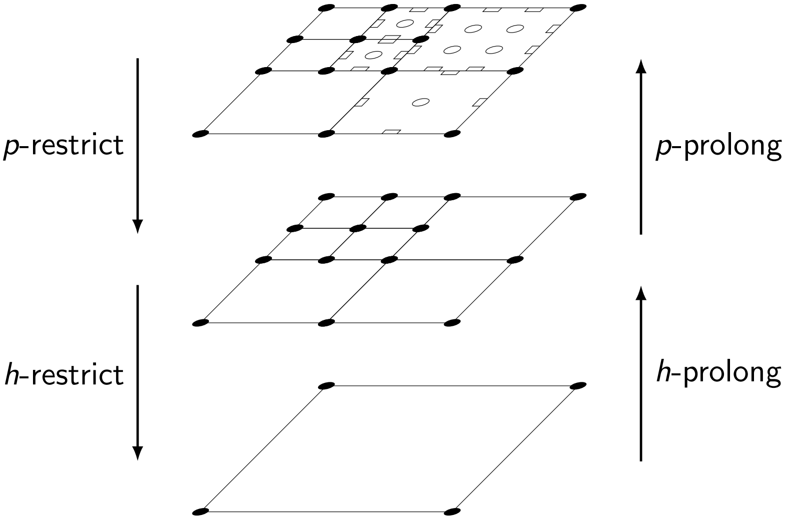
Q_1	Q_2	Q_3	Q_4
4 nodes	9 nodes	16 nodes	25 nodes

Efficient solvers

- System matrix with many nonzero entries for high-order FEs
- Number of nonzero entries per row varies with hp-FEM
- Pure algebraic multigrid (AMG) methods struggle with irregular matrices
  - Number of solver iterations explode with increasing fragmentation

- Alternative: Geometric multigrid (GMG) methods
- Hybrid-GMG: Combination of p-multigrid, h-multigrid, and AMG
- Possible with new *global coarsening* algorithms
  - `MGTransferGlobalCoarsening`

Hybrid-GMG



MatrixFree

- Memory bandwidth is bottleneck in HPC-applications
- Avoid storing system matrices
  - `MatrixFree`

- Hybrid-GMG allows us to use that feature

Parallel example

`step-75` demonstrates load balancing, hybrid-GMG, and MatrixFree methods combined in the hp-context.

Example: Laplace problem, `step-75`

- Singularity at reentrant corners for elliptic problems
- L-shaped domain:

$$\Omega = [-1,1]^2 \setminus ([0,1]\times[-1,0])$$

- Manufactured Laplace problem

$$-\nabla^2 u = 0 \quad \text{on} \quad \Omega$$
$$u = \bar{u} \quad \text{on} \quad \partial\Omega$$

$$\bar{u} = r^{2/3} \sin(2/3\,\varphi)$$
$$\|\nabla \bar{u}\| = r^{-1/3}$$

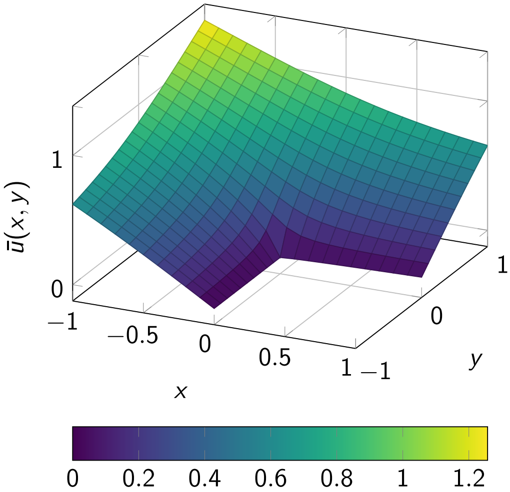


Figure: L-shaped domain

Results: Cycle 0

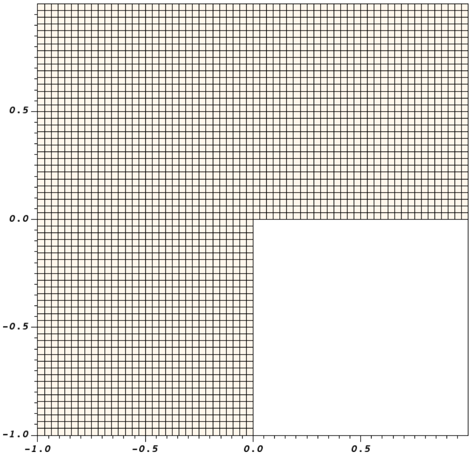


Figure: Polynomial degrees

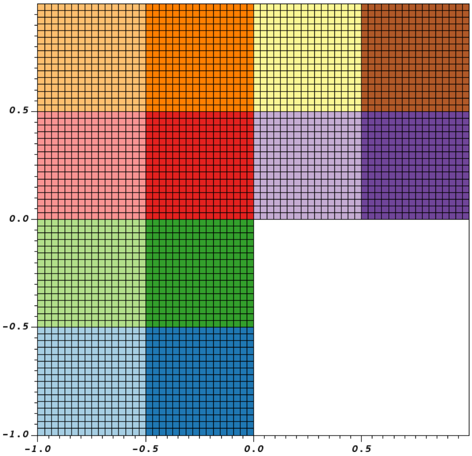


Figure: Partitioning

Results: Cycle 1

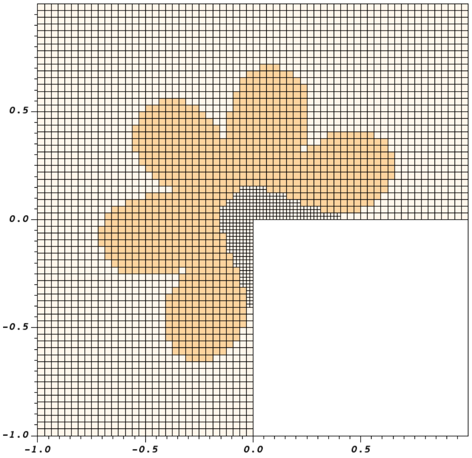


Figure: Polynomial degrees

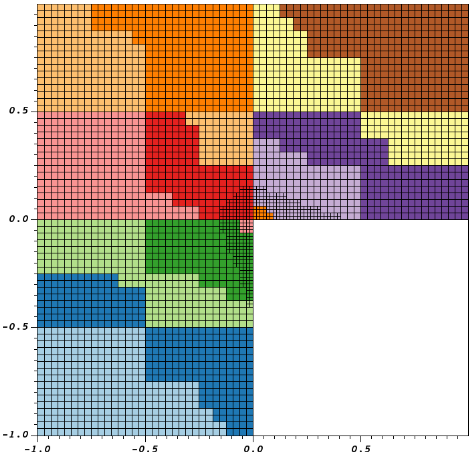


Figure: Partitioning

Results: Cycle 2

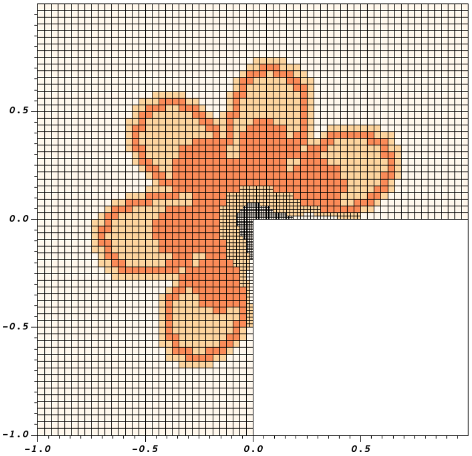


Figure: Polynomial degrees

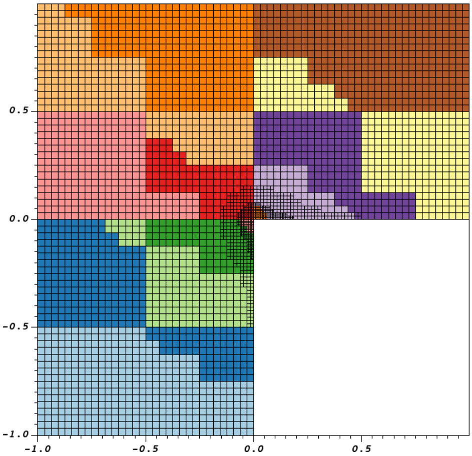


Figure: Partitioning

Results: Cycle 3

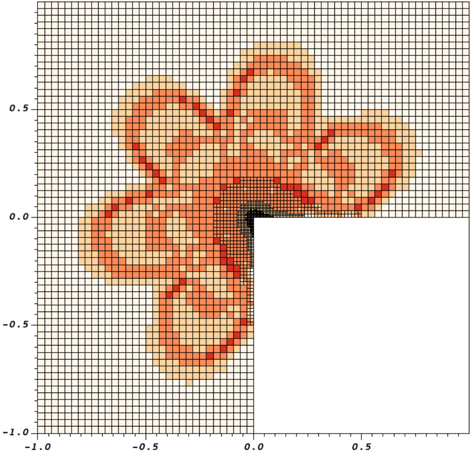


Figure: Polynomial degrees

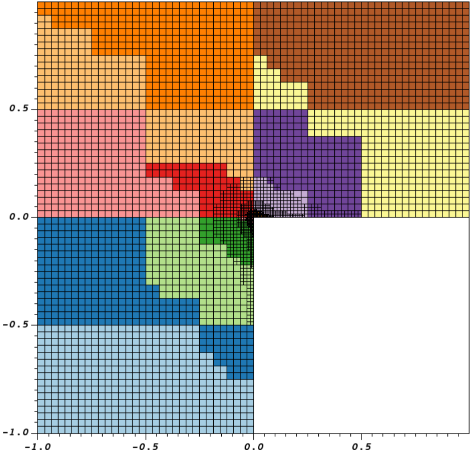


Figure: Partitioning

Results: Cycle 4

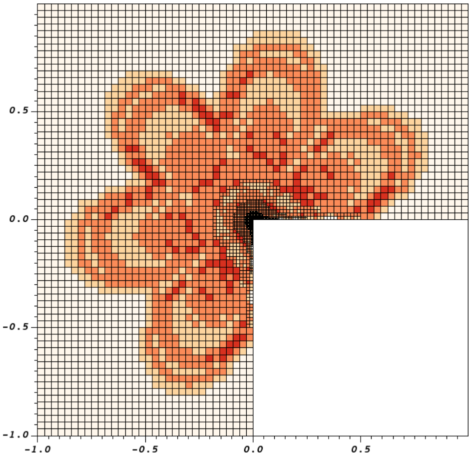


Figure: Polynomial degrees

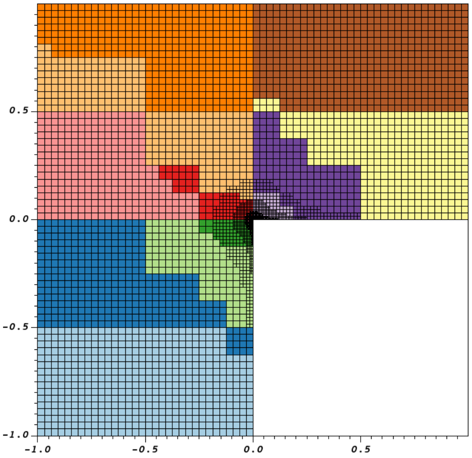


Figure: Partitioning

Results: Cycle 5

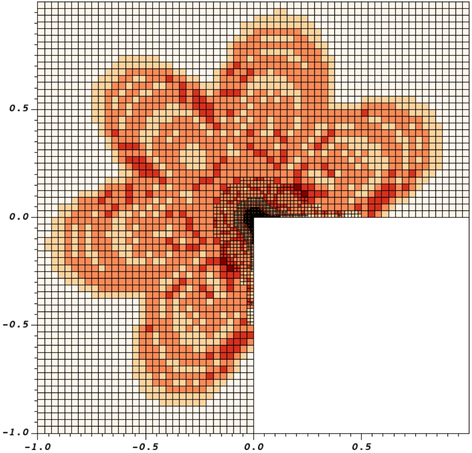


Figure: Polynomial degrees

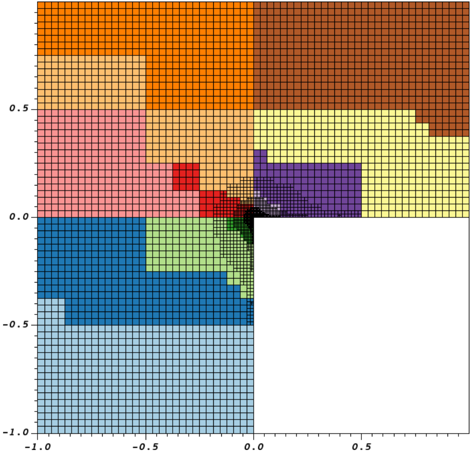


Figure: Partitioning

Results: Cycle 6

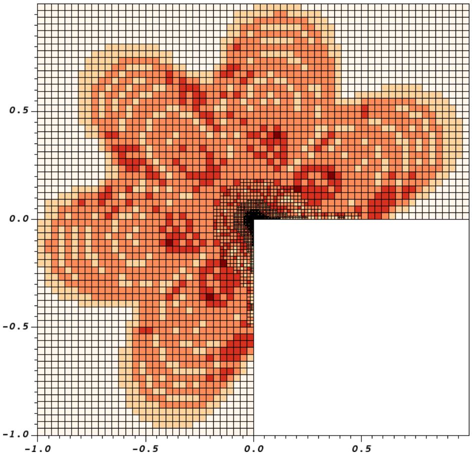


Figure: Polynomial degrees

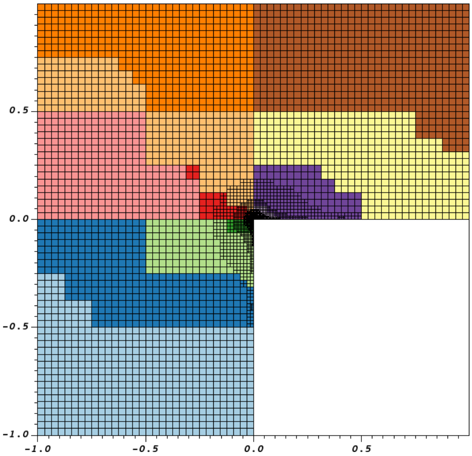


Figure: Partitioning

Results: Cycle 7

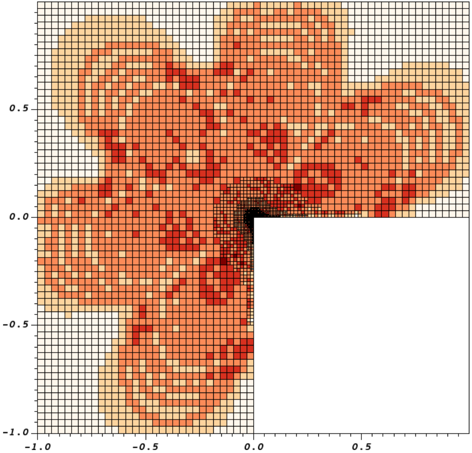


Figure: Polynomial degrees

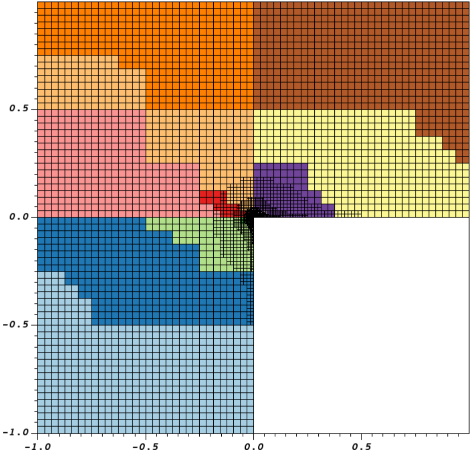


Figure: Partitioning

Results: Cycle 7 (zoom x20)

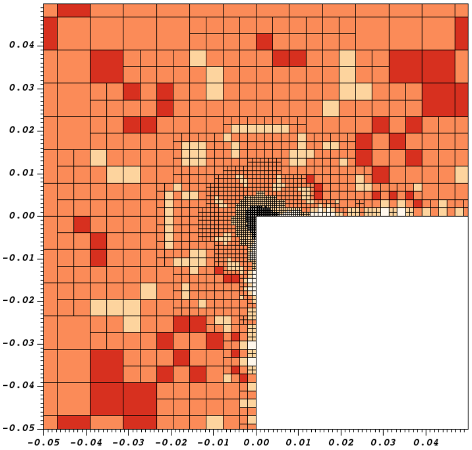


Figure: Polynomial degrees

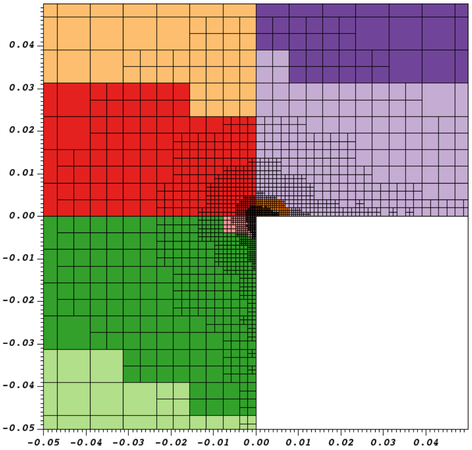


Figure: Partitioning

Results: Error convergence

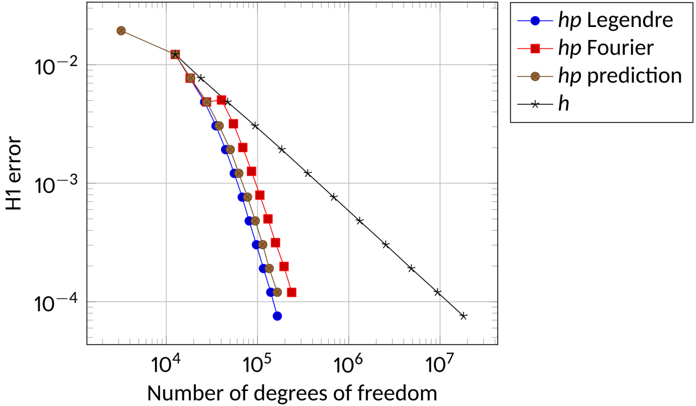


Figure: Error performances of several adaptation strategies[1]

[1] M. Fehling, Algorithms for massively parallel generic hp-adaptive finite element methods

Results: Strong scaling

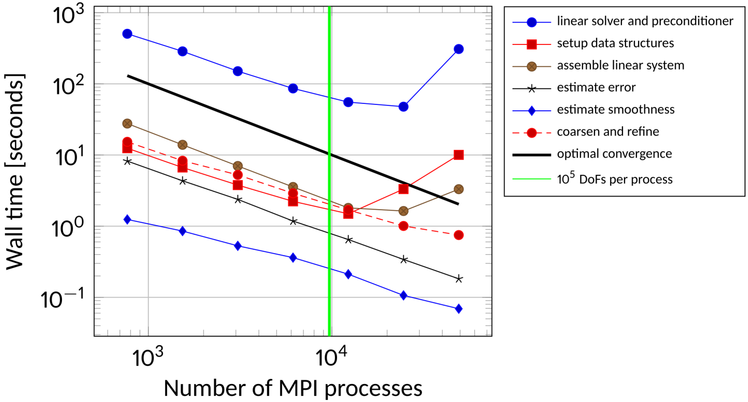


Figure: Scaling on consecutively refined meshes for 768 `MPI` processes and a pure AMG solver[1]

[1] M. Fehling, Algorithms for massively parallel generic hp-adaptive finite element methods